

FULLERENE-LIKE SPHERES WITH FACES OF NEGATIVE CURVATURE

MATHIEU DUTOIR SIKIRIĆ, MICHEL DEZA, AND MIKHAIL SHTOGRIN

ABSTRACT. Given $R \subset \mathbb{N}$, an (R, k) -sphere is a k -regular map on the sphere whose faces have gonality $i \in R$. The most interesting/useful are (geometric) *fullerenes*, i.e., $(\{5, 6\}, 3)$ -spheres.

Call $\kappa_i = 1 + \frac{i}{k} - \frac{1}{2}$ the *curvature* of i -gonal faces. (R, k) -spheres admitting $\kappa_i < 0$ are much harder to study. We consider the symmetries and construction for three new instances of such spheres: $(\{a, b\}, k)$ -spheres with $p_b \leq 3$ (they are listed), *icosahedrites* (i.e., $(\{3, 4\}, 5)$ -spheres) and, for any $c \in \mathbb{N}$, *fullerene c -disks*, i.e., $(\{5, 6, c\}, 3)$ -spheres with $p_c = 1$.

1. INTRODUCTION

Given $R \subset \mathbb{N}$, an (R, k) -sphere S is a k -regular map on the sphere whose faces have *gonalities* (numbers of sides) $i \in R$. Let v, e and $f = \sum_i p_i$ be the numbers of vertices, edges and faces of S , where p_i is the number of i -gonal faces. A graph is called *k -connected* if after removing and $k - 1$ vertices, it remains connected.

Clearly, k -regularity implies $and the *Euler formula* $2 = v - e + f$ become Gauss-Bonnet-like one $2 = \sum_i \kappa_i p_i$, where $\kappa_i = 1 + \frac{i}{k} - \frac{1}{2}$ is called (dualizing the definition in [Hi01]) the *curvature* of the i -gonal faces.$

Let $a = \min\{i \in R\}$. Then, besides the cases $k = 2$ (a -cycle) and exotic cases $a = 1, 2$, it holds

$$\frac{2k}{k-2} > a > 2 < k < \frac{2a}{a-2},$$

i.e., (a, k) should belong to the five Platonic parameter pairs $(3, 3)$, $(4, 3)$, $(3, 4)$, $(5, 3)$ or $(3, 5)$.

Call an (R, k) -sphere *standard* if $\min_{i \in R} \kappa_i = 0$, i.e., $b = \frac{2k}{k-2}$, where b denotes $\max\{i \in R\}$. Such spheres have $(b, k) = (3, 6)$, $(4, 4)$, $(6, 3)$, i.e., the three Euclidean parameter pairs. Exclusion of faces of negative curvature simplifies enumeration, while the number p_b of faces of curvature zero not being restricted, there is an infinity of such (R, k) -spheres.

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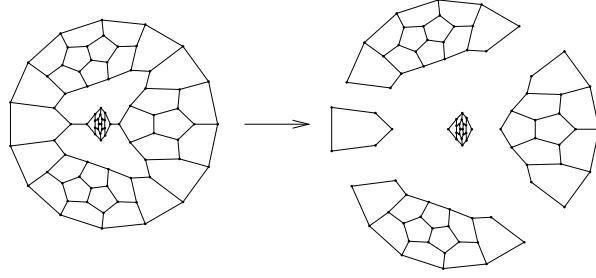


FIGURE 1. The decomposition of a $(\{5, 15\}, 3)$ -sphere seen as a $(5, 3)$ -polycycle with two, 15- and 18-gonal, holes, into elementary polycycles: two edge-split $\{5, 3\}$'s, Pen_1 , C_2 and $\{5, 3\} - e$ in the middle

An $(\{a, b\}, k)$ -sphere is an (R, k) -sphere with $R = \{a, b\}$, $1 \leq a, b$. Clearly, all possible $(a, b; k)$ for the standard $(\{a, b\}, k)$ -spheres are:

$$(5, 6; 3), (4, 6; 3), (3, 6; 3), (2, 6; 3), (3, 4; 4), (2, 4; 4), (2, 3; 6), (1, 3; 6).$$

Those eight families can be seen as spheric analogs of the regular plane partitions $\{6, 3\}$, $\{4, 4\}$, $\{3, 6\}$ with $p_a = \frac{2b}{b-a}$ a -gonal “defects” κ_a added to get the total curvature 2 of the sphere. $(\{5, 6\}, 3)$ -spheres are (geometric) *fullerenes*, important in Chemistry, while $(\{a, b\}, 4)$ -spheres are minimal projections of *alternating links*, whose components are their *central circuits* (those going only ahead) and crossings are the vertices.

We considered above eight families in [DeDu05, DeDuSh03, DHL02, DeSt03, DuDe11, DeDu11] and the book [DeDu08], where i -faces with $\kappa_i < 0$ are allowed, mainly in Chapters 15-19. Here we consider three new natural instances of $(\{a, b, c\}, k)$ -spheres, each allowing faces of negative curvature. The first Section describes such $(\{a, b\}, k)$ -spheres with $p_b \leq 3$. The second Section concerns the *icosahedrites*, i.e., $(\{3, 4\}, 5)$ -spheres, in which 4-gonal faces have $\kappa_4 = -\frac{1}{5}$. The third Section treats $(\{a, b, c\}, k)$ -spheres with $p_c = 1$, in which unique c -gonal face can be of negative curvature, especially, *fullerene c -disks*, i.e., $(\{5, 6, c\}, 3)$ -spheres with $p_c = 1$.

Note that all (R, k) -spheres with $1, 2 \notin R$ and $\kappa_i > 0$ for all $i \in R$ have $k = 4, 5$ or 3 and, respectively, $\kappa_i = \frac{1}{4}, \frac{1}{10}$ or $\kappa_i \in \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\}$. So, they are only Octahedron, Icosahedron and eleven $(\{3, 4, 5\}, 3)$ -spheres: eight dual deltahedra, Cube and its truncations on one or two opposite vertices (*Dürer octahedron*).

2. THE LIST OF $(\{a, b\}, k)$ -SPHERES WITH $p_b \leq 3$

For $a, k \geq 3$, a (a, k) -polycycle is a plane graph whose faces, besides some disjoint pairwise, including exterior one and called *holes*, are a -gons, and

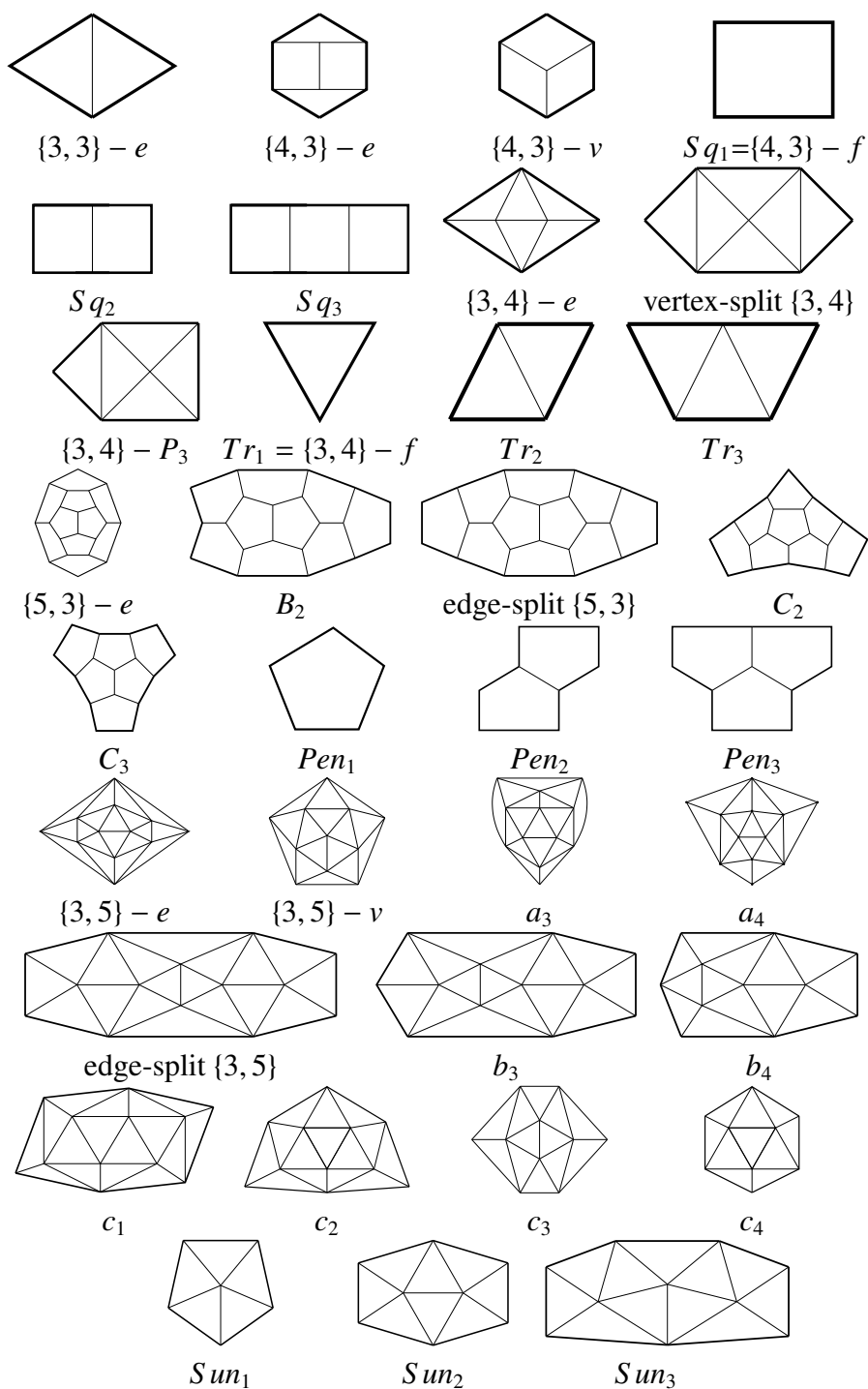


FIGURE 2. (a, k) -polycycles from Chapters 4,7 of [DeDu08] used in proofs of Theorems 2 and 3

whose vertices have degree between 2 and k with vertices not on the boundary of holes being k -valent. Let us see any $\{a, b\}$, k -sphere, after merging its adjacent b -gons in larger faces, as a (a, k) -polycycle. For a given $(\{a, b\}, k)$ -sphere, we can first remove the edges contained in two b -gons. If a vertex has a clockwise list of incident faces of the form $b^{x_1}a^{y_1} \dots b^{x_N}a^{y_N}$ with $N \geq 2$ and $x_i, y_i \geq 1$, then we split it into N different vertices. The remaining faces are a -gonal and are organized into one or more (a, k) -polycycles with the pair (a, k) being one of $(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$ and $(5, 3)$.

The $(3, 3)$ -, $(3, 4)$ - and $(4, 3)$ -polycycles are easily classified (see pages 45, 46 of [DeDu08]) and this gives a method for solving the problem of this section. For the remaining two cases, we have to introduce another method. An (a, k) -polycycle is called *elementary* if it cannot be cut along an edge into two (a, k) -polycycle. An (a, k) -polycycle admits a unique decomposition into elementary (a, k) -polycycles and the list of elementary $(5, 3)$ -, $(3, 5)$ -polycycles is given in pages 75, 76 of [DeDu08].

From the set of elementary polycycles, appearing in an $(\{a, b\}, k)$ -sphere, we can form a *decomposition graph* with the vertices being the occurring polycycles and two polycycles adjacent if they share an edge, a vertex or are connected by an edge contained in two b -gons. The connecting vertices and edges are called *active*. See in Figure 1 an example of such a decomposition.

See the list of polycycles used in this paper on Figure 2. Here Sq_i, Tr_i denote horizontal paths of i , respectively, squares and triangles, while series Pen_i, Sun_i are defined similarly. Note that $Tr_2 = \{3, 3\} - e$ and $Tr_4 = \{3, 4\} - P_4$. We define *vertex-split* $\{3, 5\}$ as unique $(3, 5)$ -polycycle obtained from a_3 by adjoining a Tr_1 , and *face-split* $\{3, 5\}$ as unique $(3, 5)$ -polycycle obtained from c_1 by adding two Tr_1 on two opposite edges.

Note that the 5-gons of unique minimal fullerene c -disk with $c = 4, 7, 8$ and $c \geq 13$ given in Figure 18 are organized, respectively, into edge-split $\{5, 3\}$, $C_3 + Pen_7$, $Pen_7 + Pen_7$ and $B_2 + Pen_{c-12} + B_2$.

Theorem 1. *There is no $(\{a, b\}, k)$ -sphere with $p_b = 1$.*

Proof. The decomposition graph of such spheres is a tree. If this tree is reduced to a vertex, then the occurring polycycle has an exterior face being a b -gon and an examination of the possibilities gives $a = b$. Otherwise, we have at least one polycycle with a unique active vertex or edge. But an inspection of the list of elementary polycycles gives that no such one satisfies the condition. \square

Note that $(\{6, 8\}, 3)$ -maps with $p_6 = 1$ exist on an oriented surface of genus 3.

Clearly, all $(\{a, b\}, k)$ -spheres with $a, k > 2$ and $p_b = 0$ are five Platonic ones denoted by $\{a, k\}$: Tetrahedron, Cube ($Prism_4$), Octahedron ($APrism_3$), Dodecahedron (snub $Prism_5$) and Icosahedron (snub $APrism_3$).

There exists unique 3-connected *trivial* $(\{a, b\}, k)$ -sphere with $p_b = 2$ for $(\{4, b\}, 3)$ -, $(\{3, b\}, 4)$ -, $(\{5, b\}, 3)$ -, $(\{3, b\}, 5)$ -: $Prism_b D_{bh}$, $APrism_b D_{bd}$, *snub* $Prism_b D_{bd}$, *snub* $APrism_b D_{bd}$, i.e., respectively, two b -gons separated by b -ring of 4-gons, $2b$ -ring of 3-gons, two b -rings of 5-gons, two $3b$ -rings of 3-gons.

Clearly, for any $t \geq 2$, there are 2-connected b -vertex $(\{2, b = 2t\}, 3)$ - and $(\{2, b\}, 2t)$ -sphere with $p_b = 2$: a circle with t disjoint 2-gons put on it and a b -gon with every edge repeated t times.

Theorem 2. *Let $b > a > 2$, $k > 2$. For any non-trivial $(\{a, b\}, k)$ -sphere with $p_b = 2$, the number $t = \frac{b}{a}$ is an integer. The list of such spheres consists of following 10 spheres for each $t \geq 2$:*

- (i) *For $(a, k) = (3, 3), (4, 3), (5, 3), (3, 4), (3, 5)$, the $(\{a, ta\}, k)$ -sphere D_{th} obtained by putting on a circle t polycycles $\{a, k\} - e$. Those polycycles are connected by an edge to their neighbors and so, only 2-connected.*
- (ii) *For $(a, k) = (3, 4), (5, 3), (3, 5)$, the $(\{a, ta\}, k)$ -sphere D_{th} obtained by partitioning of a circle into t polycycles: respectively, vertex-split $\{3, 4\}$, edge-split $\{5, 3\}$, edge-split $\{3, 5\}$.*
- (iii) *The $(\{3, 3t\}, 5)$ -spheres C_{th}, D_t , obtained by partitioning of a circle into t polycycles: respectively, vertex-split $\{3, 5\}$ ($Tr_1 + a_3$) and face-split $\{3, 5\}$ ($Tr_1 + c_1 + Tr_1$).*

Proof. If the two b -gonal faces are separated by an elementary polycycle, then we are in the case of the *snub* $Prism_b$ or *snub* $APrism_b$. Otherwise, the decomposition graph should contain one cycle separating two b -gons. Any nontrivial path connected to this cycle would have a vertex of degree 1 which we have seen to be impossible. So, the decomposition graph is reduced to this cycle. Examination of the list of polycycles with exactly two connecting edges/vertices and consideration of all possibilities gives the above list. \square

Among above spheres, only those coming from edge-split $\{5, 3\}$ edge-split $\{3, 5\}$ and face-split $\{3, 5\}$ are 3-connected. Those coming from vertex-split $\{3, 4\}$ and vertex-split $\{3, 5\}$ are 3-edge connected, but only 2-(vertex)-connected.

Let us address now the case $p_b = 3$. Note that $3 \times K_2$ with t disjoint 2-gons put on each edge is a $(\{2, b = 2 + 4t\}, 3)$ -sphere with $p_b = 3$.

Theorem 3. *Let $b \geq 2, b \neq a$ and let $(a, k) = (3, 3), (4, 3), (3, 4), (5, 3)$. Then $(\{a, b\}, k)$ -spheres with $p_b = 3$ exists if and only if $b \equiv 2, a, 2a - 2 \pmod{2a}$ and $b \equiv 4, 6 \pmod{10}$ for $a = 5$.*

Such spheres are unique if $b \not\equiv a \pmod{2a}$ and their symmetry is D_{3h} . Let $t = \lfloor \frac{b}{2a} \rfloor$. There are seven such spheres with $t = 0$ and $3 + 4 + 5 + 9$ of them for any $t \geq 1$; see corresponding Figures 4, 5, 6, 7 and detailed description below.

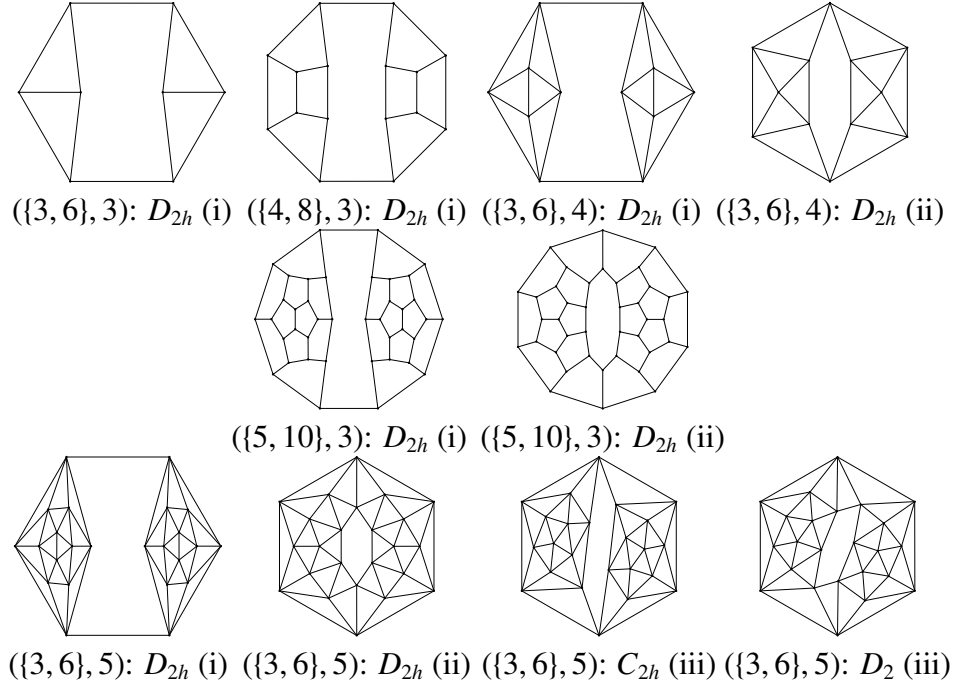


FIGURE 3. All non-trivial $(\{a, ta\}, k)$ -spheres with $p_{2a} = t = 2$; see Theorem 2

(i) The three $(\{3, b\}, 3)$ -sphere with $p_b = 3$ and $b = 2 + 6t, 4 + 6t, 3 + 6t$ come by putting t polycycles $\{3, 3\} - e$ on 3 edges of, respectively, $3 \times K_2$, Prism_3 (three 4-4 edges) and $\text{Tetrahedron } \{3, 3\}$. Only for $b = 2, 4$, the graph is 3-connected. The symmetry is C_{3v} if $b = 3 + 6t$.

(ii) All four but one $(\{4, b\}, 3)$ -spheres with $p_b = 3$ and $b = 2 + 8t, 6 + 8t, 4 + 8t$ come by putting t polycycles $\{4, 3\} - e$ on 3 edges of, respectively, $3 \times K_2$, 14-vertex $(\{4, 6\}, 3)$ -sphere (three 6-6 edges) and $\text{Cube } \{4, 3\}$ (3 incident edges). The remaining sphere is Sq_{8t-1} with its two end edges connected on each of two sides by a chain of t polycycles $\{4, 3\} - e$. This graph has symmetry C_{2v} , while other graph coming from Cube has symmetry C_{3v} . Only for $b = 6$, the graph is 3-connected.

(iii) All five but two $(\{3, b\}, 4)$ -spheres with $p_b = 3$ and $b = 2 + 6t, 4 + 6t, 3 + 6t$ come when replacing by t vertex-split $\{3, 4\}$'s, 3 vertices of, respectively, edge-doubled triangle, 9-vertex $(\{3, 4\}, 4)$ -sphere (3 vertices common to two 4-gons) and $\text{Octahedron } \{3, 4\}$ (3 vertices of a triangle). First of remaining spheres consists of Tr_{6t+5} connected to its other end on one side by a chain of t vertex-split $\{3, 4\}$'s and on the other side by a chain of t polycycles $\{3, 4\} - e$. Second remaining sphere consists of a vertex and $\{3, 4\} - P_3$.

connected by two chains of t polycycles $\{3, 4\} - e$ and one chain of t vertex-split $\{3, 4\}$'s. Those graphs have symmetry C_s , while other graph coming from Octahedron has symmetry C_{3v} . Only for $b = 2$ and $b = 4$, the graphs are 3-connected.

There are nine $(\{5, b\}, 3)$ -spheres with $p_b = 3$ for each $t = \lfloor \frac{b}{10} \rfloor \geq 1$.

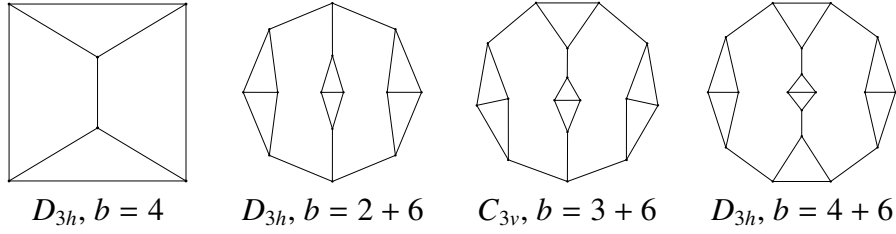
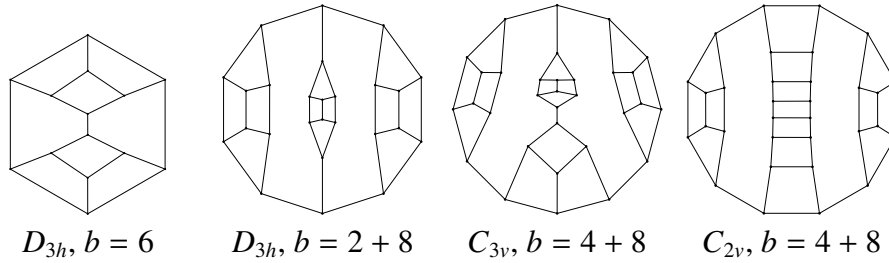
- (a) Three spheres with $b = 2 + 10t$, $b = 8 + 10t$, $b = 5 + 10t$ come by putting t polycycles $\{5, 3\} - e$ on 3 edges of $3 \times K_2$, 22-vertex $(\{5, 8\}, 3)$ -sphere (three 8-8 edges) and Dodecahedron $\{5, 3\}$ (3 incident edges). Symmetry is C_{3v} in last case and D_{3h} , otherwise. Only for $b = 8$, the graph is 3-connected.
- (b) Three 3-connected spheres with $b = 4 + 10t$, $6 + 10t$, $5 + 10t$ come by putting 3 chains of t edge-split $\{5, 3\}$'s between two polycycles, $\{Pen_3, Pen_3\}$, $\{C_3, C_3\}$ and $\{Pen_3, C_3\}$. Symmetry is C_{3v} in last case and D_{3h} , otherwise.
- (c) Three 2-connected $(\{5, 5 + 10t\}, 3)$ -spheres of symmetry C_s come from, respectively:
 - c1) a chain $B_2 + t$ times edge-split $\{5, 3\}$ connected on both ends by t times $\{5, 3\} - e$;
 - c2) $C_2 + t(\{5, 3\} - e) + Pen_1$ connected on both ends by t times edge-split $\{5, 3\}$;
 - c3) Pen_{10t+9} connected on one side by t times edge-split $\{5, 3\}$ and on the other one by t times $\{5, 3\} - e$.

Proof. It is not possible to have an elementary polycycle separating three b -gonal faces. Hence, the decomposition graph has three faces and is either formed of two vertices of degree 3 connected by chains of vertices of degree 2, or a vertex of degree 4 connected by two chains of vertices of degree 2 on each side. An examination of the possibilities along the same lines gives the above result. \square

Note that all $(\{a, b\}, k)$ -spheres with $p_b = 3$ and symmetry $\neq C_s$, are bR_j (i.e., each b -gon has exactly j edges of adjacency with b -gons); see *face-regularity* in Section 3). $j = 2 \lceil \frac{b}{2a} \rceil$ for $(a, k) = (3, 3)$, $(4, 3)$ and $j = 0$ for $(a, k) = (3, 4)$, $(3, 5)$. For $(a, k) = (5, 3)$, we have $j = 0$ or $2 \lceil \frac{b}{2a} \rceil$.

In case $(a, k) = (3, 5)$, we have 17 infinite series of spheres but no proof that the list is complete. See below the list of $(\{3, b\}, 5)$ -spheres obtained and in Figure 8, 9 their pictures for small b . All but (a) have $b = 3 + 6t$. All but (a) and (d_2) are only 2-connected. By R_e , V_{sp} , E_{sp} , and F_{sp} we denote a chain of t polycycles A with A being $\{3, 5\} - e$, vertex-split $\{3, 5\}$, edge-split $\{3, 5\}$ and face-split $\{3, 5\}$, respectively.

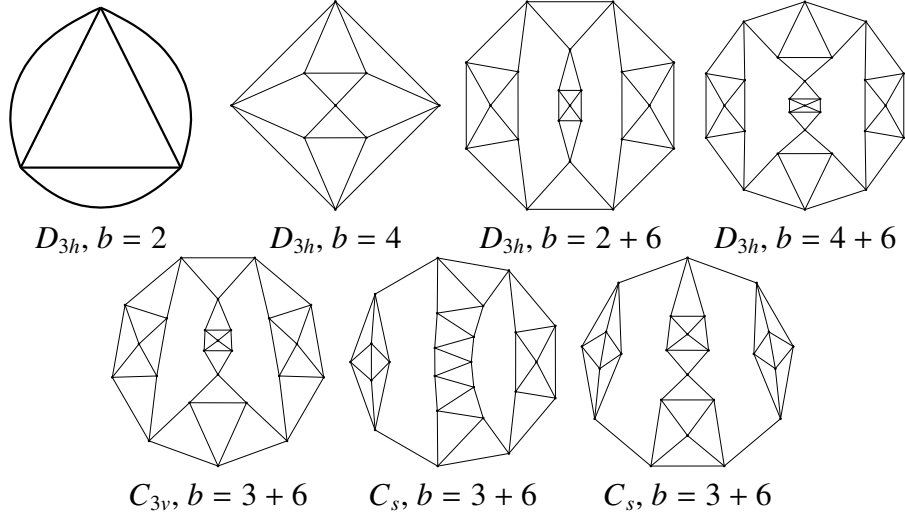
- (a) Three spheres with $b = 2 + 6t$, $4 + 6t$, $3 + 6t$ obtained by putting three E_{sp} between two polycycles, $\{Tr_1 + 3Tr_1, Tr_1 + 3Tr_1\}$, $\{c_4 +$

FIGURE 4. All $(\{3, b\}, 3)$ -spheres with $p_b = 3$ for $2 \leq b \leq 10$ FIGURE 5. All $(\{4, b\}, 3)$ -spheres with $p_b = 3$ for $2 \leq b \leq 12$

$3Tr_1, c_4 + 3Tr_1\}$ and $\{c_4 + 3Tr_1, Tr_1 + 3Tr_1\}$. Symmetry is C_3 in last case and D_3 , otherwise.

- (b₁) C_1 : $c_1 + Tr_1 + F_{sp}$ with ends connected by R_e and V_{sp} .
- (b₂) C_s : Sun_{6t+5} with ends connected by R_e and E_{sp} .
- (b₃) C_1 : $b_3 + Tr_1 + E_{sp}$ with ends connected by R_e and F_{sp} .
- (b₄) C_s : $b_3 + 2Tr_1 + E_{sp}$ with ends connected by V_{sp} and E_{sp} .
- (b₅) C_1 : $Sun_{4+6t} + 3Tr_1$ with ends connected by V_{sp} and F_{sp} .
- (c₁) C_s : a vertex and $(\{3, 5\} - v) + 2Tr_1$ connected by R_e and two V_{sp} 's.
- (c₂) C_s : a vertex and a_3 connected by two R_e 's and one V_{sp} .
- (c₃) C_1 : a vertex and $c_2 + 2Tr_1$ connected by two V_{sp} 's and one F_{sp} .
- (d₁) C_s : Sun_1 and $Sun_3 + Tr_1$ connected by V_{sp} and two E_{sp} 's.
- (d₂) C_s : Sun_2 and $Sun_1 + 2Tr_1$ and connected by E_{sp} and two F_{sp} 's.
- (e₁) C_{3v} : Tr_1 and a_4 connected by three V_{sp} 's.
- (e₂) C_s : Tr_1 and c_2 connected by R_e and two F_{sp} 's.
- (e₃) C_s : $2Tr_1$ and $c_3 + 3Tr_1$ connected by V_{sp} and two F_{sp} 's.
- (e₄) C_1 : $3Tr_1$ and $b_4 + Tr_1$ connected by V_{sp} , E_{sp} and F_{sp} .

Note that all 15 orbits of $(\{3, 3 + 6t\}, 5)$ -spheres constructed were built from the 15 orbits of triples of triangles in Icosahedron. Note also that $\binom{20}{3} = \sum_G \frac{120}{|G|} = 5 \times 120 + 8 \times 60 + 40 + 20$, since there are 5, 8, 1, 1 cases with symmetry C_1, C_s, C_3, C_{3v} , respectively. There may be other spheres in this case and for other values of b .

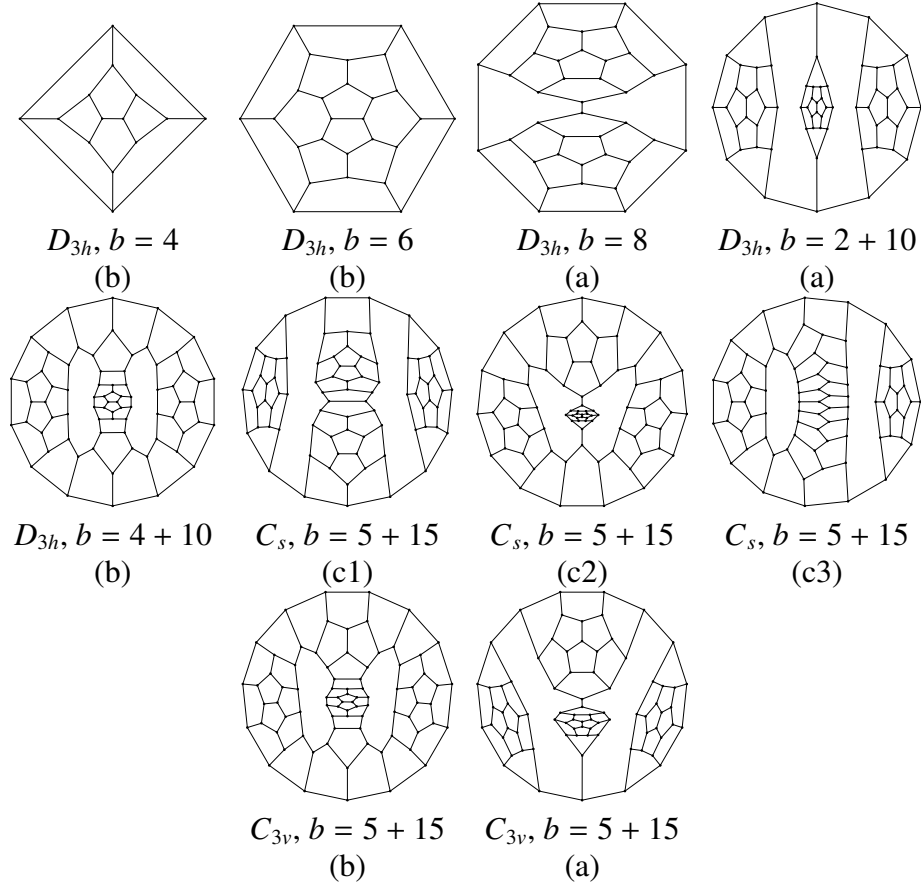
FIGURE 6. All $(\{3, b\}, 4)$ -spheres with $p_b = 3$ for $2 \leq b \leq 14$

3. ICOSAHEDRITES

We call *icosahedrite* the $(\{3, 4\}, 5)$ -spheres. They are in a sense the simplest 5-valent plane graphs. Clearly, for them it holds $p_3 = 20 + 2p_4$ and $v = 12 + 2p_4$. Note that all $(\{a, 3\}, 5)$ -spheres with $a < 3$ are: $(\{1, 3\}, 5)$ -sphere with $(p_1, p_3; v) = (2, 6; 4)$ and four $(\{2, 3\}, 5)$ -spheres with $(p_2, p_3; v) = (4, 4; 4)$ (two), $(3, 8; 6)$, $(2, 12; 8)$. Remaining $(\{1, 2, 3\}, 5)$ -spheres should have $(p_1, p_2, p_3; v) = (1, 3, 1; 2)$, $(2, 1, 2; 2)$, $(1, 2, 5; 4)$ or $(1, 1, 9; 6)$; only 2nd and 3rd exist.

The simplest icosahedrite is Icosahedron, which is a $(\{3\}, 5)$ -sphere of symmetry I_h . One way to obtain icosahedrite is from an *octahedrite*, i.e. a $(\{3, 4\}, 4)$ -sphere G . To every vertex of G we associate a square, to every edge (coherently) a pair of adjacent triangles and faces are preserved. Only the rotational symmetries of G are preserved in the final icosahedrite $C(G)$. If one applies C to Octahedron, then one gets the smallest (24 vertices) icosahedrite of symmetry O (see Figure 14). Applying it to the infinite regular plane tiling $\{4, 4\}$ by squares, one gets the Archimedean *snub square tiling* (3.3.4.3.4). Note that there is only one other *infinite Archimedean icosahedrite*, i.e., a vertex-transitive 5-valent tiling of the plane by regular 3- and 4-gons only: *elongated triangular tiling* (3.3.3.4.4).

For a given icosahedrite, a *weak zigzag WZ* is a circuit of edges such that one alternate between the left and right way but never extreme left or right. The usual *zigzag* is a circuit such that one alternate between the extreme left and extreme right way.

FIGURE 7. All $(\{5, b\}, 3)$ -spheres with $p_b = 3$ for $2 \leq b \leq 15$

A zigzag or weak zigzag is called *edge-simple*, respectively *vertex-simple*, if any edge, respectively vertex, of it occurs only once. A vertex-simple zigzag is also edge-simple. If WZ is vertex-simple of length l , then one can construct another icosahedrite with l more vertices.

Clearly, the weak zigzags (as well as the usual zigzags) doubly cover the edge set. For example, 30 edges of Icosahedron are doubly covered by 10 weak vertex-simple zigzags of length 6, as well as by 6 usual vertex-simple zigzags of length 10. Clearly, a 3-gon surrounded by 9 3-gons or a 4-gon surrounded by 12 3-gons corresponds to weak zigzags of length six or eight, respectively. In fact, no other vertex-simple weak zigzags exist.

In Table 1 we list the number of v -vertex icosahedrites for $v \leq 32$. From this list it appears likely that any icosahedrite is 3-connected.

Theorem 4. *A $(\{3, 4\}, 5)$ -sphere exist if v is even, $v \geq 12$ and $v \neq 14$. Their number grows at least exponentially with v .*

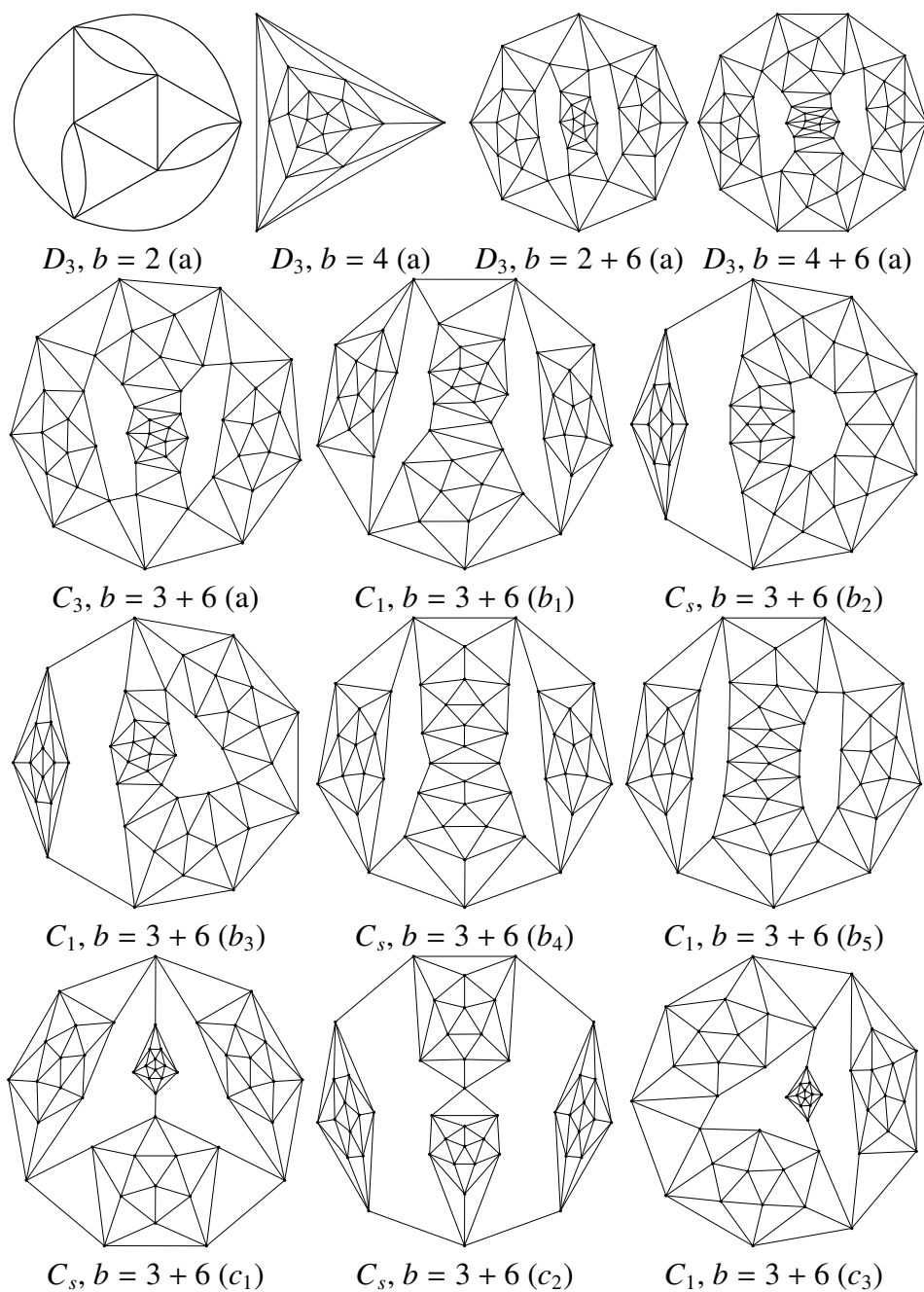


FIGURE 8. All known $(\{3, b\}, 5)$ -spheres with $p_b = 3$ and $2 \leq b \leq 10$ (part 1)

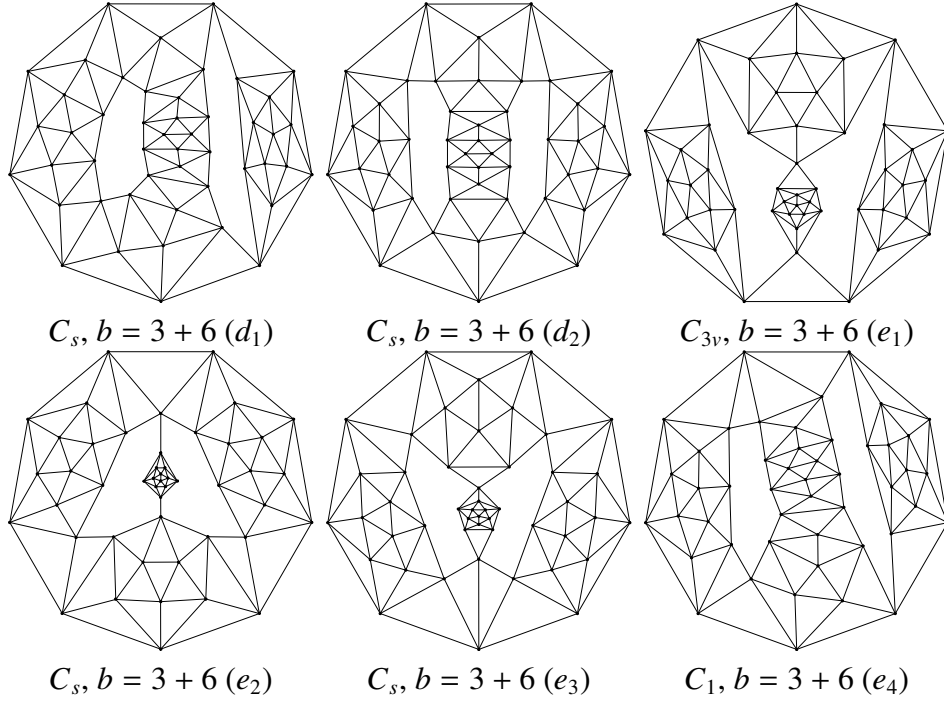


FIGURE 9. All known $(\{3, b\}, 5)$ -spheres with $p_b = 3$ and $2 \leq b \leq 10$ (part 2)

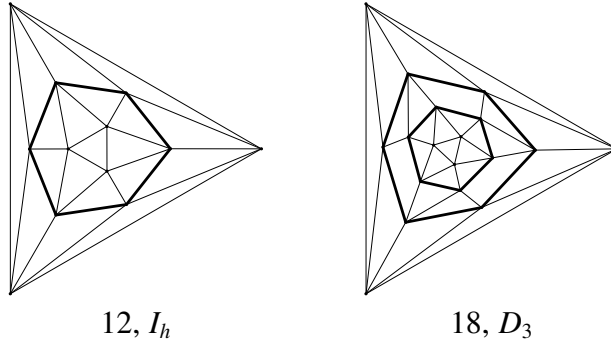


FIGURE 10. A simple weak zigzag in an icosahedrite and the resulting expansion

TABLE 1. The number of v -vertex icosahedrites

v	12	14	16	18	20	22	24	26	28	30	32
Nr	1	0	1	1	5	12	63	246	1395	7668	45460

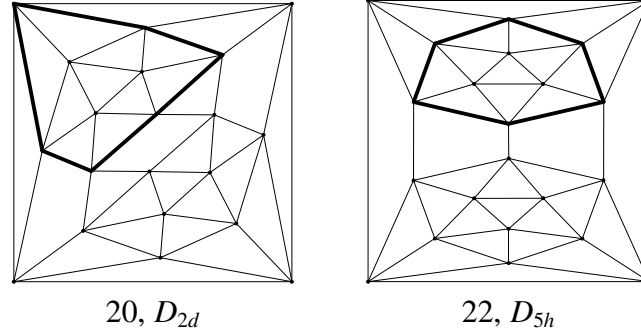


FIGURE 11. Two icosahedrites having a simple weak zigzag of length 6

Proof. If there is an weak zigzag of length 6 in an icosahedrite G , then we can insert a *corona* (6-ring of three 4-gons alternated by three pairs of adjacent 3-gons) instead of it and get an icosahedrite with 6 more vertices. Since there are such icosahedrite for $n = 18, 20, 22$ (see Figure 10, 11), we can generate the required graphs. There is always two options when inserting the corona and so the number grows exponentially as required. \square

In the case of fullerenes or other standard $(\{a, b\}, k)$ -spheres, the number of a -gons is fixed and the structure is made up of patches of b -gons. The parametrization of graphs, generalizing one in [Thur98], including the case of one complex parameter - *Goldberg-Coxeter construction* ([DuDe04]) - are thus built. A consequence of this is the polynomial growth of the number of such spheres. This does not happen for the case of icosahedrites and their parametrization, if any, looks elusive.

As a consequence of this increased freedom, we can more easily build a new icosahedrite from a given one by an expansion operation, while in the case of standard $(\{a, b\}, k)$ -spheres, one is essentially restricted to the Goldberg-Coxeter construction. The operation A , respectively A' , replaces each vertex of an icosahedrite G by 6, respectively 21, vertices and gives icosahedrites $A(G)$, $A'(G)$ with the same symmetries as G (see Figure 12). Moreover, for any $m \geq 2$ we can define an operation that replaces every face F by a patch and add $m - 1$ rings of squares and pairs of triangles. The operation B_2 on a 4-gon is shown in Figure 12. The resulting map $B_m(G)$ has only the rotational symmetries of G and associate to every vertex of G $1 + 5m(m - 1)$ vertices in $B_m(G)$.

Theorem 5. Any symmetry group of icosahedrites if one of following 38:

$C_1, C_i, C_s, S_4, S_6, S_8, S_{10}, C_2, C_{2h}, C_{2v}, C_3, C_{3h}, C_{3v}, C_4, C_{4h}, C_{4v}, C_5, C_{5h}, C_{5v}, D_2, D_{2h}, D_{2d}, D_3, D_{3h}, D_{3d}, D_4, D_{4h}, D_{4d}, D_5, D_{5h}, D_{5d}, O, O_h, T, T_d, T_h, I, I_h.$

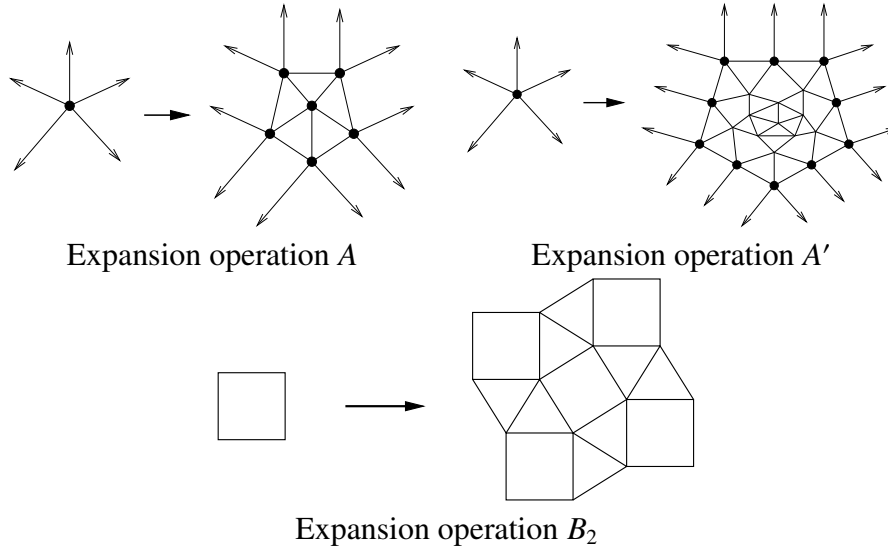


FIGURE 12. Three expansion operations on icosahedrites

Proof. By the face sizes and vertex sizes, the list of possibilities is the one indicated. We used the enumeration up to 32 vertices to find many groups and their minimal representative. If a sphere has a 3-fold axis, then it necessarily passes through two 3-gons. Those two 3-gons can be replaced by 4-gons and ipso facto we get examples of a sphere with 4-fold axis. From this we got the icosahedrites with groups C_4 , C_{4h} , C_{4v} , D_4 , D_{4h} , D_{4d} and S_8 . All but 40-vertex one C_4 , C_{4v} , D_{4h} are minimal.

Now, suppose that a 2-fold axis of rotation passes by two edges, which are both contained in two triangles. Then we insert a vertex on those two edges and replace the 2-fold symmetry by a 5-fold symmetry. By iterating over all known isocahedrites, and all such 2-fold axis, we get the symmetries C_5 , C_{5h} , C_{5v} , S_{10} and D_{5h} , D_{5d} , D_5 ; last three are minimal. For the cases of O_h , T , T_d and T_h , we obtained examples by hand drawing. The 132 vertices icosahedrite of symmetry I in Figure 14 is obtained from Isocahedron by operation B_2 . The minimal known (actually, examples are minimal whenever $v \leq 32$) are given in Figure 13, 14.

Aggregating groups $\mathbf{C}_1 = \{C_1, C_s, C_i\}$, $\mathbf{C}_m = \{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D}_m = \{D_m, D_{mh}, D_{md}\}$, $\mathbf{T} = \{T, T_d, T_h\}$, $\mathbf{O} = \{O, O_h\}$, $\mathbf{I} = \{I, I_h\}$, all 38 symmetries of $(\{3, 4\}, 5)$ -spheres are: \mathbf{C}_1 , \mathbf{C}_m , \mathbf{D}_m for $2 \leq m \leq 5$ and \mathbf{T} , \mathbf{O} , \mathbf{I} . 5-, 4- and 3-fold symmetry exists if and only if, respectively, $p_4 \equiv 0 \pmod{5}$ (i.e., $v = 2p_4 + 12 \equiv 2 \pmod{10}$), $p_4 \equiv 2 \pmod{4}$ (i.e., $v = 2p_4 + 12 \equiv 0 \pmod{8}$) and $p_4 \equiv 0 \pmod{3}$ (i.e., $v = 2p_4 + 12 \equiv 0 \pmod{6}$).

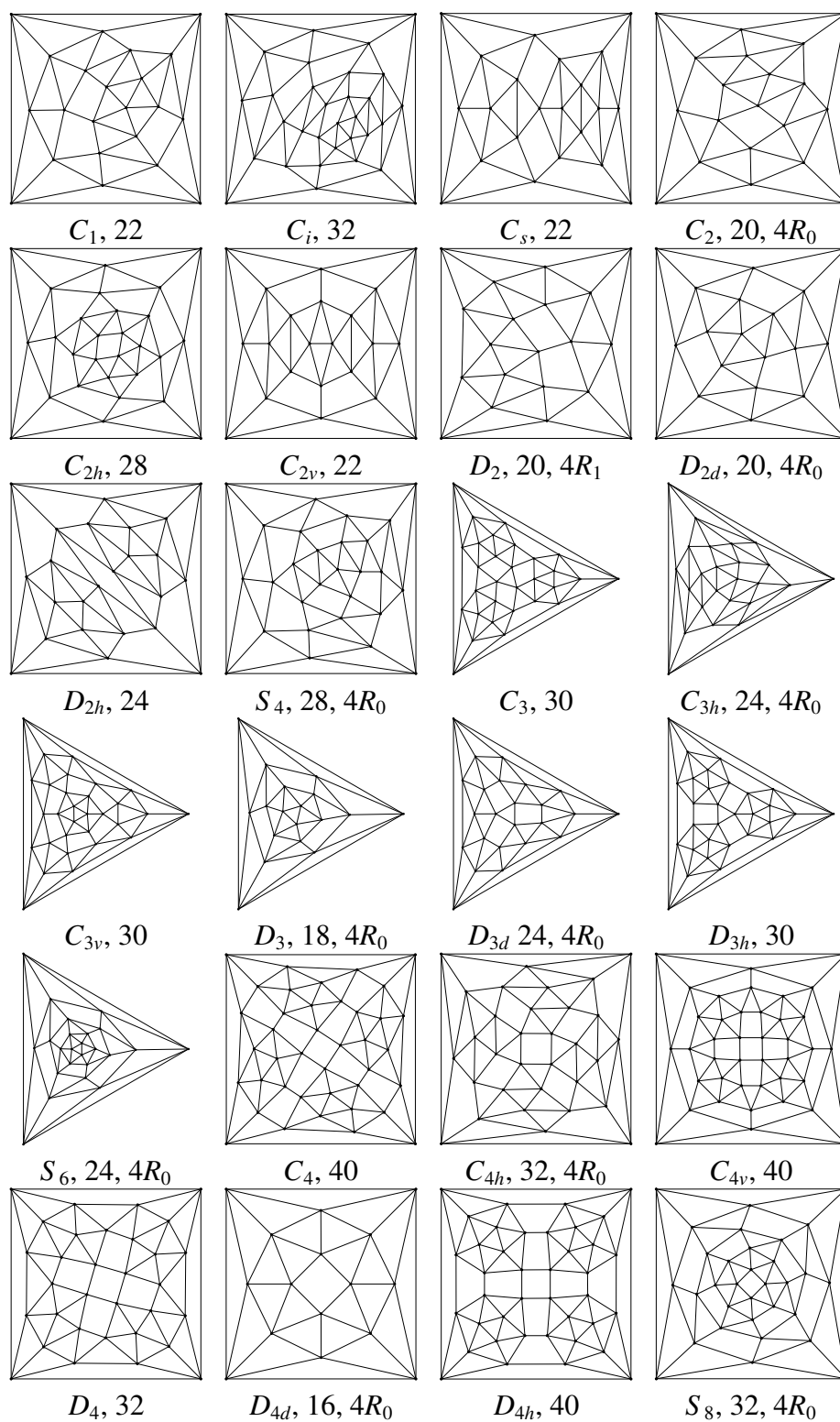


FIGURE 13. Examples of icosahedrites for all possible symmetry groups (part 1); all 21 with at most 32 vertices are minimal ones

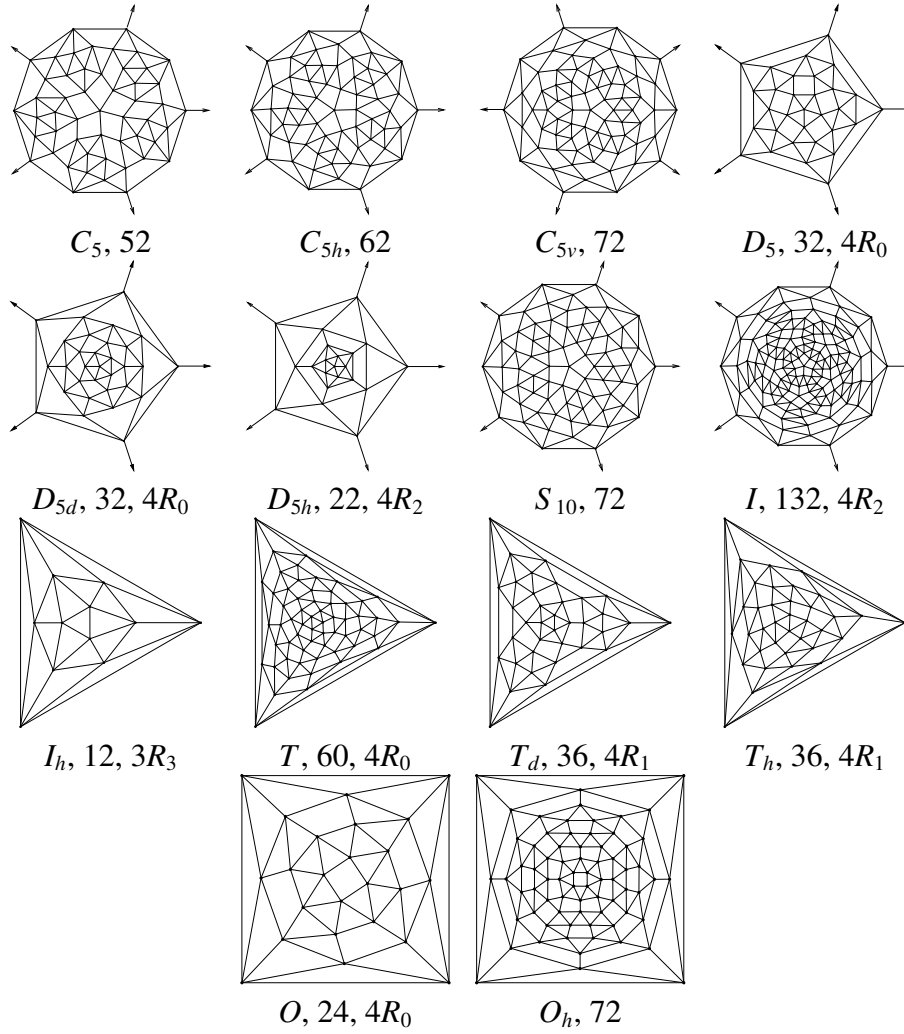


FIGURE 14. Examples of icosahedrites for all possible symmetry groups (part 2); all 5 with at most 32 vertices are minimal ones

Any group appear an infinite number of times since, for example, one gets an infinity by applying construction A iteratively.

From the above result it appears that the only limitations for the group are coming from the rotation axis. It seems possible that this also occurs for all $(\{a, b\}, k)$ -spheres with b -gons being of negative curvature.

A map is said to be *face-regular*, or, specifically, pR_i if every face of size p is adjacent to exactly i faces of the same size p .

Theorem 6. (i) *The only icosahedrite which is $3R_i$ is Icosahedron which is $3R_3$*

(ii) *For $i = 0, 1$ and 2 there is an infinity of icosahedrites that are $4R_i$.*

Proof. Let N_{ij} denote the number of i - j edges, i.e., those which are common to an i -gon and an j -gon; so, $e = N_{33} + N_{34} + N_{44}$. But $N_{34} + N_{44} \leq 4p_4$ with equality if and only if our icosahedrite is $4R_0$. So,

$$2e = 3p_3 + 4p_4 = p_3 + 2(20 + 2p_4) + 4p_4 \leq 2N_{33} + 8p_4,$$

i.e., $2N_{33} \geq p_3 + 40$. It excludes the cases $3R_0$ and $3R_1$, since then $2N_{33} = 0$ and p_3 , respectively.

If our icosahedrite is $3R_2$, then $N_{33} + N_{34} = p_3 + p_3$, implying $N_{44} = p_4 - 10$. Any $4R_0$ icosahedrite with 32 vertices (i.e., with $p_4 = 10$) has $\frac{N_{33}}{p_3} = 2$, i.e., it is $3R_2$ in average. Now, $3R_2$ means that the 3-gons are organized in rings separated by 4-gons. Such rings can be either five 3-gons with common vertex, or 12 3-gons around a 4-gon, or $APrism_m$, $m > 4$. In each case, such ring should be completed to an icosahedrite by extruding edge to keep 5-regularity. In order to be isolated from other 3-gons, the ring should have twice longer ring of 4-gons around it. The faces touching the ring of 3-gons in a vertex only, could not be 3-gonal since they have two 4-gonal neighbors. So, the isolating ring of 4-gons consists of 4-gons adjacent (to the ring of 3-gons) alternated by 4-gons touching it only in a vertex. Easy to see that such process cannot be closed.

The number of icosahedrites $4R_0$ is infinite; such series can be obtained by the operation A (see Figure 12) from, say, unique 16-vertex icosahedrite. An infinity of $4R_1$ icosahedrites can be obtained in the following way. We take the $4R_1$ icosahedrite of Figure 15 and the ring bounded by two overlined circuits can be transformed into any number of concentrated rings. An infinity of icosahedrites $4R_2$ can be obtained from the 22-vertices icosahedrite of symmetry D_{5h} in Figure 14. It suffices to add layers of five 4-gons alternated by layers of ten 3-gons (as in Archimedean elongated triangular plane tiling (3.3.3.4.4)). \square

Among all 54,851 icosahedrites with $v \leq 32$, there are no $4R_3$ and only four specimens $4R_2$. Those with $4R_3$ should have $2N_{33} = 3p_3 - p_4$ and v divisible by 4, but we doubt they exist. Note that part (i) of Theorem 6 can be easily generalized on $(\{3, b\}, 5)$ -sphere with any $b \geq 4$.

All icosahedrites with only edge-simple usual zigzags and $v \leq 32$ are three of those seven (see Figure 16) having only edge-simple weak zigzags: 12, I_h with (10^6) , 24, D_{3d} with $(6, 10^6, 18^3)$ and 28, D_{2h} with $(10^6, 20^4)$. We expect that any icosahedrite with only edge-simple usual or weak zigzags, if such ones exist for $v \geq 34$, has v divisible by four. Also, for $v \leq 32$, the

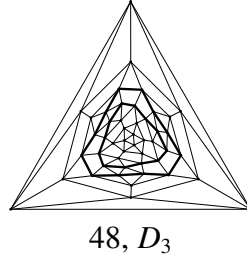


FIGURE 15. The icosahedrite used in proof of Theorem 6 as the first one in an infinite series with $4R_1$

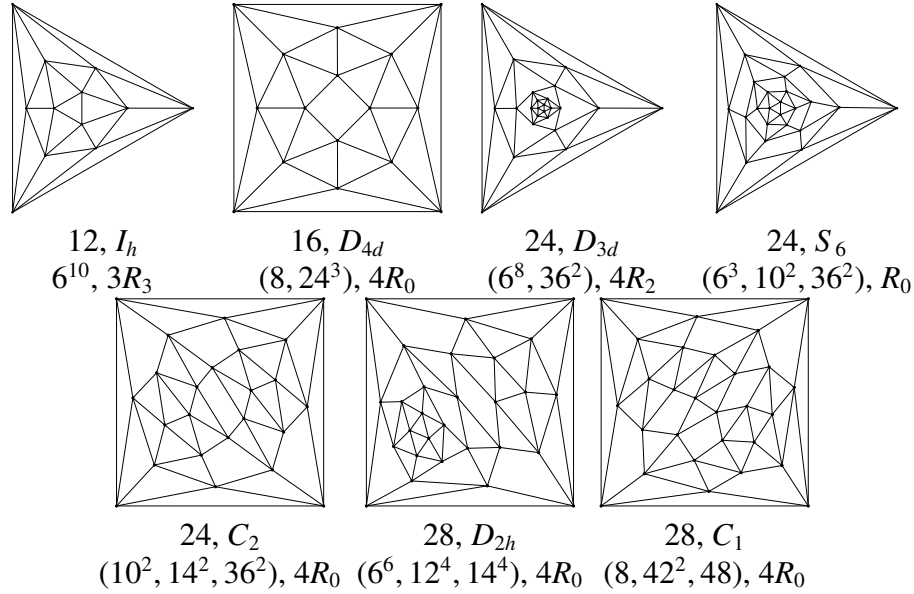


FIGURE 16. The icosahedrites with only edge-simple weak zigzags and at most 32 vertices

maximal number of zigzags and weak zigzags are realized only by edge-simple ones whenever they exist.

Snub $APrism_b$, Snub Cube and Snub Dodecahedron with $(v, b) = (4b, b)$, $(24, 4)$ and $(60, 5)$, respectively, are bR_0 and only known b -gon-transitive $(\{3, b\}, 5)$ -spheres. They are also, besides Icosahedron, only known $(\{3, b\}, 5)$ -spheres with at most two orbits of 3-gons. The Archimedean $(\{3, b\}, 5)$ -plane tilings $(3.3.4.3.4)$, $(3.3.3.4.4)$, $(3.3.3.3.6)$ have $b = 4, 4, 6$, respectively. They are transitive on b -gons and vertices. They are also, respectively, $(4R_1, 3R_1)$, $(4R_2, 3R_2)$, $6R_0$ and have 1, 1, 2 orbits of 3-gons.

4. ON $(\{a, b, c\}, k)$ -SPHERES WITH $p_c = 1$

Clearly, an $(\{a, b, c\}, k)$ -sphere with $p_c = 1$ has

$$v = \frac{2}{k-2}(p_a - 1 + p_b) = \frac{2}{2k - a(k-2)}(a + c + p_b(b-a))$$

vertices and (setting $b' = \frac{2k}{k-2}$) $p_a = \frac{b'+c}{b'-a} + p_b \frac{b-b'}{b'-a}$ a -gons. So, $p_a = \frac{b+c}{b-a}$ if $b = b'$, i.e., $(\{a, b\}, k)$ -sphere is standard.

We are especially interested in *fullerene c -disks*, i.e., $(\{5, 6, c\}, 3)$ -spheres with $p_c = 1$. It exists for any $c \geq 1$ and has $p_5 = c + 6$, $v = 2(p_6 + c + 5)$. Clearly, there is an infinity of fullerene c -disks for any $c \geq 1$.

The only way to get fullerene 1-disk, is to get a $(\{5, 6, r, s\}, 3)$ -sphere with $p_r = p_s = 1$, $(r, s) = (3, 4), (3, 3), (2, 3), (2, 4)$ and add 4 vertices: 2-gon on the $r - s$ edge and then erect an edge with 1-gon from the middle of $2 - r$ edge. In fact, only $(3, 4)$ is possible and minimal such graph has 36 vertices, proving minimality of 40-vertex 1-disk.

One can check that all $(\{a, b, c\}, 3)$ -spheres with $p_c = 1$, $2 \leq a, b, c \leq 6$ and $c \neq a, b$ are four series with $(a, b, c) = (4, 6, 2), (5, 6, 2), (5, 6, 3), (5, 6, 4)$ and following two 10-vertex spheres of symmetry C_{3v} having $p_a = p_b = 3$, $p_c = 1$: $(\{4, 5, 3\}, 3)$ - (Cube truncated on one vertex) and $(\{3, 5, 6\}, 3)$ - (Tetrahedron truncated on three vertices). There are also series of $(\{1, 4, 2\}, 4)$ - and $(\{3, 4, 2\}, 4)$ -spheres with $p_2 = 1$.

Theorem 1 implies that $(\{a, b, c\}, k)$ -sphere with $p_c = 1$ and $p_b = 0$ has $c = a$, i.e., it is the k -regular map $\{a, k\}$ on the sphere. We conjecture that a $(\{a, b, c\}, k)$ -sphere with $p_c = p_b = 1 < a, c$ has $c = b$, i.e., it is a $(\{a, b\}, k)$ -sphere with $p_b = 2$. Note that the 2-vertex $(\{1, 4, 2\}, 4)$ -sphere has $p_2 = p_4 = 1 = a < c < b$.

Call $(\{a, b, c\}, k)$ -*thimble* any $(\{a, b, c\}, k)$ -sphere with $p_c = 1$ such that the c -gon is adjacent only to a -gons. *Fullerene c -thimble* is the case $(a, b, c; k) = (5, 6, c; 3)$ of above. It exists if and only if $c \geq 5$.

Moreover, we conjecture that for odd or even $c \geq 5$, the following $(5c-5)$ - or $(5c-6)$ -vertex c -thimble is a minimal one; it holds for $5 \leq c \leq 10$ since this construction generalizes cases 7, 9-1 and 8, 10-3 in Figure 18, as well as cases $c = 5$ and 6, of minimal c -disks. Take the c -ring of 5-gons, then then put inside a concentric c -ring of 5- or 6-gons: 3-path of 5-gons and, on opposite side, 3-path or 3-ring, for even or odd c , of 5-gons. Remaining $c-6$ or $c-5$ 5- or 6-gons of interior c -ring are 6-gons. Finally, fill inside of the interior c -ring by the $\frac{c-4}{2}$ - or $\frac{c-5}{2}$ -path of 6-gons. Another generalization of the minimal 6-disk is, for $c = 6t$, $\frac{c(c+18)}{6}$ -vertex c -thimble of symmetry (for $c > 6$) C_6 or C_{6v} . In fact, take $6t$ -ring of 5-gons, then, inside of it, $6t$ -ring, where six equispaced 5-gons are alternated by $(t-1)$ -tuples of 6-gons. Finally, put inside, for $i = t-1, t-2, \dots, 1$, consecutively $6i$ -rings of 6-gons.

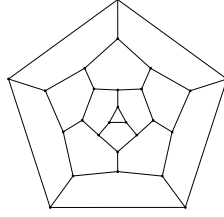


FIGURE 17. Minimal 3-disk of type $(6, 6, 5)$; it is also C_s -minimal 3-disk

Any such c -thimble can be elongated by adding an outside ring 5-gons along the c -gon and transforming inside ring of 5-gons along it into a ring of 6-gons. Let a $(\{5, 6, c\}, 3)$ -sphere with $p_c = 1$ have a simple *zigzag* (left-right circuit without self-intersections). A *railroad* is a circuit of 6-gons, each of which is adjacent to its neighbors on opposite edges. Let us elongate above sphere by a railroad along zigzag and then let us cut elongated sphere in the middle of this ring. We will get two c -thimbles.

Theorem 7. *All simple zigzags of any elongated c -thimble are parallel and its railroads are parallel c -rings of 6-gons, forming a cylinder.*

In fact, suppose that there is a railroad in an c -thimble not belonging to the cylinder of parallel c -rings of 6-gons along the boundary c -ring of 5-gons.

Let us cut it in the middle. The thimble is separated into an smaller c -thimble with $p_5 = c$ and a $(\{5, 6, c\}, 3)$ -sphere with $p_5 = 6$. But this sphere can not contain at least $c + 1$ of $c + 6$ original 5-gons: c boundary 5-gons and, at other side of the cylinder, at least one 5-gon. Only a railroad belonging to the cylinder can go around this 5-gon. \square

Given a c -disk, call its *type*, the sequence of gonalitys of its consecutive neighbors. So, any c -thimble has type $(5, \dots, 5)$.

There are bijections between v -vertex 1-disks, $(v - 2)$ -vertex 2-disks of type $(5, 6)$ and $(v - 4)$ -vertex $(\{5, 6, 3, 4\}, 3)$ -spheres with unique and adjacent 3 and 4-gon. Using it, we found the minimal 1-disk, given in Figure 18. There is a bijection between v -vertex 3-disks of type $(6, 6, 6)$ and $(v - 2)$ -vertex fullerenes with $(5, 5, 5)$ -vertex (collapse the 3-gon). There is a bijection between v -vertex 3-disks of type $(5, 6, 6)$ and $(v - 2)$ -vertex fullerenes with $(5, 5, 6)$ -vertex (delete the edge adjacent to 5-gon). So, the possible types (and minimal number of vertices for examples) are: (6) ($v = 40$) for 1-disks; $(6, 6)$ ($v = 26$), $(6, 5)$ ($v = 40 - 2$) for 2-disks; $(6, 6, 6)$ ($v = 22$), $(6, 5, 5)$ ($v = 40 - 6$), $(6, 6, 5)$ ($v = 26$, see Figure 17) for 3-disks.

One can see unique $(\{5, 6\}, 3)$ -spheres with $v = 20, 24$ as minimal fullerene c -disks with $c = 5, 6$. Clearly, the minimal fullerene 3- and 4-disks are 22-vertex spheres obtained from Dodecahedron by truncation on a vertex or an edge. As well, C_1 -minimal 3-disk and C_2 -minimal 4-disk are truncations of 28-vertex (D_2) or 24-vertex $(\{5, 6\}, 3)$ -sphere on, respectively, a vertex or an edge.

Conjecture 1. *An v -vertex fullerene c -disk with $c \geq 1$, except the cases $(c, v) = (1, 42), (3, 24)$ and $(5, 22)$, exists if and only if v is even and $v \geq 2(p_6(c) + c + 5)$. Here $p_6(c)$ denotes the minimal possible number of 6-gons in a fullerene c -disk.*

We have $p(1) = 14, p(2) = 6, p(3) = 3, p(4) = 2, p(5) = 0, p(6) = 1, p(7) = 3, p(8) = 4, p(9) = 6, p(10) = 7, p(11) = 8, p(12) = 5$, and $p(c) = 6$ for $c \geq 13$. For $2 \leq c \leq 20$, the conjecture was checked by computation and all minimal fullerene c -disks (2, 3, 10 for $c = 9, 10, 11$ and unique, otherwise) are listed; see Figure 18 for $1 \leq c \leq 14, c \neq 5, 6$. All c -disks there having $c \geq 3$, except three among 11-disks, are 3-connected.

The c -pentatube is $2(c + 11)$ -vertex fullerene c -disk of symmetry C_2, C_s for even, odd c , respectively. Its 5-gons are organized in two $(5, 3)$ -polycycles B_2 separated by Pen_{c-12} and its six 6-gons are organized into two 3-rings each shielding a B_2 from Pen_{c-12} . The c -pentatube is unique minimal for $13 \leq c \leq 20$; we expect that it remains so for any $c \geq 13$.

Theorem 8. *The possible symmetry groups of a $(\{5, 6, c\}, 3)$ -sphere with $p_c = 1$ and $c \neq 5, 6$ are C_n, C_{nv} with $n \in \{1, 2, 3, 5, 6\}$ and n dividing c .*

In fact, any symmetry of a $(\{5, 6, c\}, 3)$ -sphere should stabilize unique c -gon. So, the possible groups are only C_n and C_{nv} with n ($1 \leq n \leq c$) dividing c . Moreover, $n \in \{1, 2, 3, 5, 6\}$ since, on the axis has to pass by a vertex, edge or face. Remind that $C_s = C_{1v}$. \square

Cases (xi) and (xvi) in [DDFo09] show that the possible symmetry groups (minimal v) of a fullerene c -disk with $c = 3$ and $c = 4$ are C_1 (30), C_s (26), C_3 (34), C_{3v} (22) and C_1 (28), C_s (24), C_2 (26), C_{2v} (22). Minimal examples are given there. For $c = 1, 2, 7, 8, 9$ see minimal examples on Figure 19. For $c = 2$, those examples are all of type (6, 6) and coming from one-edge truncation of $(\{5, 6\}, 3)$ -spheres with, respectively, 28 (D_2), 26, 24 and 20 vertices.

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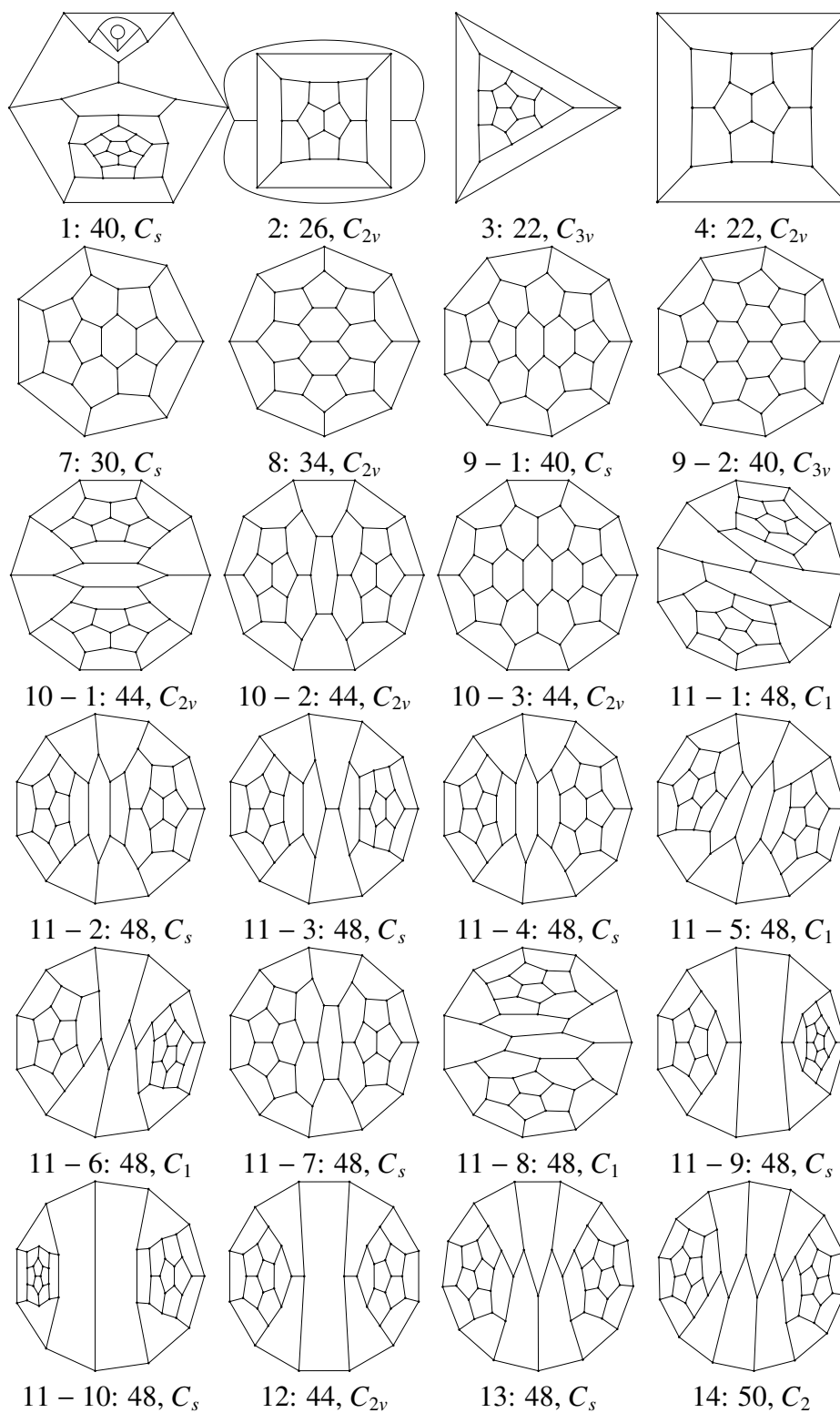


FIGURE 18. Minimal fullerene c -disks with $1 \leq c \leq 14$, $c \neq 5, 6$

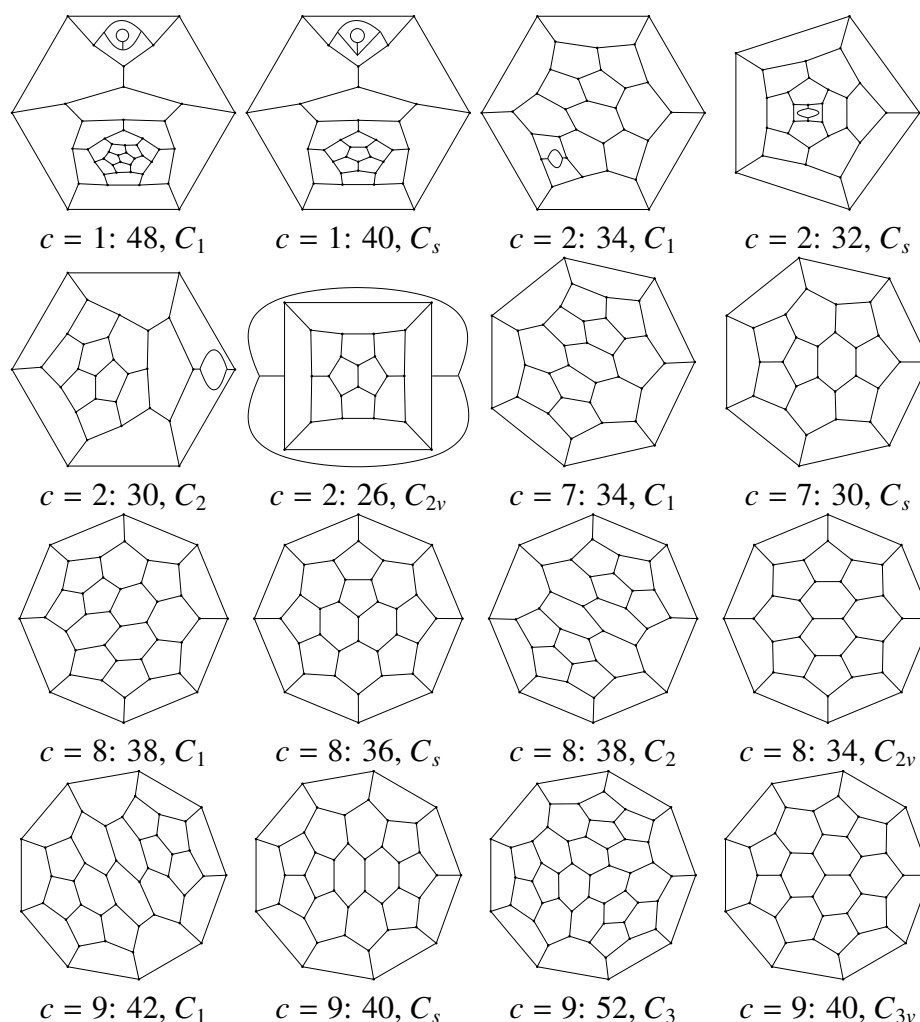


FIGURE 19. Minimal fullerene c -disks for each possible group with $c = 1, 2, 7, 8$, and 9

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MATHIEU DUTOUR SIKIRIĆ, RUDJER BOSKOVIĆ INSTITUTE, BIJENICKA 54, 10000 ZAGREB, CROATIA

E-mail address: mdsikir@irb.hr

MICHEL DEZA, M. DEZA, ÉCOLE NORMALE SUPÉRIEURE, PARIS

E-mail address: Michel.Deza@ens.fr

MIKHAIL SHTOGRIN, STEKLOV MATHEMATICAL INSTITUTE, MOSCOW

E-mail address: stogrin@mi.ras.ru